

The Weight Distribution of a Class of Cyclic Codes Related to Hermitian Forms Graphs

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Abstract—The determination of weight distribution of cyclic codes involves evaluation of Gauss sums and exponential sums. Despite of some cases where a neat expression is available, the computation is generally rather complicated. In this note, we determine the weight distribution of a class of reducible cyclic codes whose dual codes may have arbitrarily many zeros. This goal is achieved by building an unexpected connection between the corresponding exponential sums and the spectrums of Hermitian forms graphs.

Index Terms—Cyclic codes, Cayley graphs, Hermitian forms graphs, weight distribution.

I. INTRODUCTION

For a cyclic code \mathcal{C} of length l over finite field \mathbb{F}_p with p prime, let A_i be the number of codewords in \mathcal{C} of Hamming weight i . The weight distribution $\{A_0, A_1, \dots, A_l\}$ is an important research subject in coding theory. Let $h(x)$ be the parity check polynomial of \mathcal{C} . We say that \mathcal{C} is irreducible (resp. reducible) if $h(x)$ is irreducible (resp. reducible) over \mathbb{F}_p . When $h(x) = h_0(x)h_1(x) \cdots h_{s-1}(x)$ for some distinct irreducible polynomials $h_i(x)$ over \mathbb{F}_p , the code \mathcal{C} is the dual of a cyclic code with s zeros.

An identity due to McEliece [15] shows that weights of irreducible cyclic codes can be expressed via Gauss sums. So the determination of the weights of irreducible cyclic codes can be tackled using number theoretic techniques (see [7], [15], [16], [22], [26]). However, this problem is extremely difficult in general since the same is true for the evaluation of Gauss sums. When an irreducible cyclic code has exactly one nonzero weight, a nice characterization has been given in [4], [23], [24]. Besides, the class of two-weight irreducible cyclic codes was extensively studied. The necessary and sufficient conditions for an irreducible cyclic code to have at most two weights were given by Schmidt and White [19]. And they conjectured that all irreducible two-weight cyclic codes consist of two infinite families and eleven sporadic examples. The reader can get more information on the weight distribution of irreducible cyclic codes in [4].

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For reducible cyclic codes, the determination of weight distribution involves evaluation of exponential sums. Despite of some cases where a neat expression is available (see [3], [5], [8], [10], [11], [12], [13], [14], [18], [27]), the computation is generally rather complicated. Although delicate techniques were applied to the computation, most of these literature, to our knowledge, can only obtain the weight distribution of reducible cyclic codes whose dual codes have two or three zeros. The exponential sums which have been explicitly evaluated seem to share a common feature that they attain only a few distinct values.

In this paper, we determine the weight distribution of a class of reducible cyclic codes whose dual codes may have arbitrarily many zeros. This goal is achieved by building a surprising connection between the involved exponential sums and the spectrums of Hermitian forms graphs. The rest of this paper is organized as follows. The codes we considered will be introduced in Section II. A brief introduction to Cayley graphs and Hermitian forms graphs is given in Section III. We build the connection between exponential sums and spectrums of Hermitian forms graphs in Section IV. After presenting this connection, the weight distribution follows immediately. A brief conclusion will be given in the last section.

II. THE CODE $\mathcal{C}_{(p,m)}$

First we fix some notation. Let p be a prime and $q = p^n$ with $n = 2m$, where m is odd. Write $t = (m-1)/2$. Suppose π is a primitive element of \mathbb{F}_q . Let $h_0(x)$ be the minimal polynomial of $\pi^{-(p^m+1)}$ over \mathbb{F}_p . Then $\deg h_0(x) = m$. Let $h_i(x)$ be the minimal polynomial of $\pi^{-(p^{2i-1}+1)}$ over \mathbb{F}_p , where $1 \leq i \leq t$. For any integer $l > 1$ with $l|2m$, we have $\pi^{-(p^{2i-1}+1)(p^{2m/l}-1)} \neq 1$, where $1 \leq i \leq t$. Thus $\deg h_i(x) = n$ for $1 \leq i \leq t$. Since for $1 \leq i < j \leq t$, there does not exist any positive integer k such that

$$p^k(p^{2i-1}+1) \equiv p^{2j-1}+1 \pmod{q-1},$$

the elements $\pi^{-(p^{2i-1}+1)}$ and $\pi^{-(p^{2j-1}+1)}$ have distinct minimal polynomials over \mathbb{F}_p . So the polynomials $h_i(x)$ are distinct for $0 \leq i \leq t$.

Let $\mathcal{C}_{(p,m)}$ be the cyclic code with parity check polynomial $h_0(x)h_1(x) \cdots h_t(x)$ over \mathbb{F}_p . Then the code $\mathcal{C}_{(p,m)}$ is the dual of a cyclic code with $t+1$ zeros and $\dim_{\mathbb{F}_p} \mathcal{C}_{(p,m)} = m^2$. Let Tr_i^j denote the trace mapping from \mathbb{F}_{p^j} to \mathbb{F}_{p^i} . The codewords in $\mathcal{C}_{(p,m)}$ can be expressed as

$$\mathbf{c}_{[\alpha_0, \alpha_1, \dots, \alpha_t]} = (c_0, c_1, \dots, c_{q-2}) \quad (\alpha_0 \in \mathbb{F}_{p^m}, \alpha_1, \dots, \alpha_t \in \mathbb{F}_q)$$

where

$$c_i = \text{Tr}_1^m(\alpha_0 \pi^{i(p^m+1)}) + \sum_{j=1}^t \text{Tr}_1^n(\alpha_j \pi^{i(p^{2j-1}+1)}),$$

for $0 \leq i \leq q-2$ (see [2]). Hence the Hamming weight of the codeword $\mathbf{c}_{[\alpha_0, \alpha_1, \dots, \alpha_t]}$ is

$$w_H(\mathbf{c}) = p^{n-1}(p-1) - \frac{1}{p} \sum_{a \in \mathbb{F}_p^*} T(a\alpha_0, a\alpha_1, \dots, a\alpha_t),$$

where

$$T(\alpha_0, \alpha_1, \dots, \alpha_t) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^m(\alpha_0 x^{p^m+1}) + \sum_{j=1}^t \text{Tr}_1^n(\alpha_j x^{p^{2j-1}+1})}.$$

Generally speaking, it is very difficult to obtain the value distribution of $T(\alpha_0, \alpha_1, \dots, \alpha_t)$ for $\alpha_0 \in \mathbb{F}_{p^m}$, $\alpha_1, \dots, \alpha_t \in \mathbb{F}_q$, especially when t is large. In the next section, we will build a surprising connection between the multiset $\{T(\alpha_0, \alpha_1, \dots, \alpha_t) \mid \alpha_0 \in \mathbb{F}_{p^m}, \alpha_1, \dots, \alpha_t \in \mathbb{F}_q\}$ and the eigenvalues of Hermitian forms graphs, which can reduce the complexity of computation for $T(\alpha_0, \alpha_1, \dots, \alpha_t)$ remarkably.

III. CAYLEY GRAPHS AND HERMITIAN FORMS GRAPHS

Now we record some known results on Cayley graphs and Hermitian forms graphs.

A. Cayley graphs

Let G be a finite group and $D \subset G$ be a subset. The Cayley graph $\text{Cay}(G, D)$ on G with connection set D is the directed graph with vertex set G and edge set $\{(g, h) \mid g, h \in G, hg^{-1} \in D\}$.

Define $D^{(-1)} = \{d^{-1} \mid d \in D\}$. Then $\text{Cay}(G, D)$ is undirected if $D = D^{(-1)}$. Furthermore, $\text{Cay}(G, D)$ is k -regular with $k = |D|$. If G is a finite abelian group, it is easy to compute the spectrum of $\text{Cay}(G, D)$. For any character χ of G , define $\chi(D) = \sum_{d \in D} \chi(d)$. The character group of G is denoted by \hat{G} , with $|\hat{G}| = |G|$.

Lemma 3.1: Let $\Gamma = \text{Cay}(G, D)$ be a Cayley graph on a finite abelian group G with connection set D . Suppose $A = A(\Gamma)$ is the adjacency matrix of Γ . Then each character χ of G corresponds to an eigenvector of A with eigenvalue $\chi(D)$. In particular, the spectrum of Γ is the multiset $\{\chi(D) \mid \chi \in \hat{G}\}$.

Proof: Let χ be a character of G . Let e_χ be the column vector $(\chi(g))_{g \in G}$. For any $h \in G$, we have

$$(Ae_\chi)_h = \sum_{g \sim h} \chi(g) = \left(\sum_{d \in D} \chi(d) \right) \chi(h) = \chi(D) \chi(h).$$

Hence, e_χ is an eigenvector of A with eigenvalue $\chi(D)$. All characters in \hat{G} give rise to $|G|$ linearly independent eigenvectors, thus one obtains the spectrum of Cayley graph Γ via the character group \hat{G} . ■

B. Hermitian forms graphs

Let $V = \mathbb{F}_{r^2}^d$, where r is a prime power. For any $x \in \mathbb{F}_{r^2}$, its conjugate \bar{x} is defined by $\bar{x} = x^r$. A matrix H over \mathbb{F}_{r^2} is called Hermitian if $H = H^*$, where H^* is the conjugate transpose of H . Let \mathcal{H} denote the abelian group formed by all $d \times d$ Hermitian matrices over \mathbb{F}_{r^2} under the matrix addition. Clearly, we have $|\mathcal{H}| = r^{d^2}$. The *Hermitian forms graph* on V is the graph whose vertices are the elements of \mathcal{H} and in which $H_1, H_2 \in \mathcal{H}$ are adjacent whenever $\text{rank}(H_1 - H_2) = 1$. Equivalently, the Hermitian forms graph is the Cayley graph $\text{Cay}(\mathcal{H}, \mathcal{D})$, where $\mathcal{D} = \{H \in \mathcal{H} \mid \text{rank}(H) = 1\}$. A $d \times d$ Hermitian matrix H of rank 1 can be written as $H = \mathbf{a}^T \bar{\mathbf{a}}$, where $\mathbf{a} = (a_1, \dots, a_d) \in V$ and $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_d)$. Since for any $\mathbf{a}, \mathbf{b} \in V$, $\mathbf{a}^T \bar{\mathbf{a}} = \mathbf{b}^T \bar{\mathbf{b}}$ if and only if $\mathbf{a} = \gamma \mathbf{b}$ for some $(r+1)$ -th root of unity γ , we have $|\mathcal{D}| = (r^{2d} - 1)/(r+1)$.

It is well known that the Hermitian forms graph on V is a distance regular graph with classical parameters $(d, b, \alpha, \beta) = (d, -r, -r-1, -(-r)^d - 1)$ [1, Table 6.1]. The eigenvalues of the Hermitian forms graph were first computed by Stanton [21]. Here we quote the more accessible formulas given in [1]. For any integers $j \geq i \geq 0$ and $b \neq 0, 1$, the Gaussian binomial coefficients with basis b are defined by

$$\begin{bmatrix} j \\ i \end{bmatrix}_b = \begin{cases} \prod_{l=0}^{i-1} \frac{b^j - b^l}{b^i - b^l} & \text{if } i \geq 1, \\ 1 & \text{if } i = 0. \end{cases}$$

Lemma 3.2: [1, Corollary 8.4.4] Let $V = \mathbb{F}_{r^2}^d$, where r is a prime power. The Hermitian forms graph defined on V has eigenvalues

$$\theta_0 = \frac{r^{2d} - 1}{r + 1}, \quad \theta_j = \frac{r^{2d} - 1}{r + 1} + (-r)^{2d-j} \begin{bmatrix} j \\ 1 \end{bmatrix}_{(-r)},$$

for $1 \leq j \leq d$. Their corresponding multiplicities are

$$f_0 = 1, \quad f_j = \begin{bmatrix} d \\ j \end{bmatrix}_{(-r)} \prod_{l=0}^{j-1} [(-1)^{d+1} r^d + (-1)^{l+1} r^l],$$

where $1 \leq j \leq d$.

IV. THE WEIGHT DISTRIBUTION OF THE CODE $\mathcal{C}_{(p,m)}$

Throughout this section, p, q, n, m, t are defined as in Section II. Consider the abelian group

$$\mathcal{G} = \mathbb{F}_{p^m} \times \underbrace{\mathbb{F}_q \times \mathbb{F}_q \times \dots \times \mathbb{F}_q}_t,$$

and its subset

$$\mathcal{S} = \{(x^{p^m+1}, x^{p+1}, x^{p^3+1}, \dots, x^{p^{m-2}+1}) \mid x \in \mathbb{F}_q^*\}.$$

It is easy to see that $|\mathcal{S}| = (q-1)/(p+1)$. Let $W = \mathbb{F}_{p^2}^m$ and \mathcal{H} be the abelian group consisting of all $m \times m$ Hermitian matrices over \mathbb{F}_{p^2} . Let $\mathcal{D} = \{H \in \mathcal{H} \mid \text{rank}(H) = 1\}$. Clearly, the Hermitian forms graph on W is the Cayley graph $\text{Cay}(\mathcal{H}, \mathcal{D})$. The following lemma shows that the Cayley graph $\text{Cay}(\mathcal{G}, \mathcal{S})$ shares the same spectrum with $\text{Cay}(\mathcal{H}, \mathcal{D})$.

Lemma 4.1: For odd m , the Hermitian forms graph Γ_1 on $W = \mathbb{F}_{p^2}^m$ is isomorphic to the Cayley graph $\Gamma_2 = \text{Cay}(\mathcal{G}, \mathcal{S})$. In particular, Γ_1 and Γ_2 have the same spectrum.

Proof: Since Γ_1 is just the Cayley graph $\text{Cay}(\mathcal{H}, \mathcal{D})$, the result will follow immediately if we can find a group isomorphism φ from \mathcal{H} to \mathcal{G} satisfying that $\varphi(\mathcal{D}) = \mathcal{S}$.

Let e_1, \dots, e_m be a basis of \mathbb{F}_q over \mathbb{F}_{p^2} and $\mathbf{e} = (e_1, e_2, \dots, e_m)$. For any $H \in \mathcal{H}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^m$, we define $f_H(\mathbf{x}, \mathbf{y}) = \mathbf{x}H\mathbf{y}^T$, where \mathbf{y}^T is the transpose of \mathbf{y} . Now we construct a mapping φ from \mathcal{H} to \mathcal{G} by sending $H \in \mathcal{H}$ to

$$\varphi(H) = (f_H(\mathbf{e}, \mathbf{e}^{p^m}), f_H(\mathbf{e}, \mathbf{e}^{p^p}), f_H(\mathbf{e}, \mathbf{e}^{p^3}), \dots, f_H(\mathbf{e}, \mathbf{e}^{p^{m-2}})),$$

where $\mathbf{e}^s := (e_1^s, e_2^s, \dots, e_m^s)$ for any integer s . It is straightforward to verify that $\varphi(H) \in \mathcal{G}$ and φ is a group homomorphism.

Now we want to show that φ is an isomorphism. First, we prove that φ is injective. For a matrix $H = (h_{ij})$ and an integer s , we denote $H^s = (h_{ij}^s)$. Suppose $\varphi(H) = (0, 0, \dots, 0)$, i.e., $\mathbf{e}H(\mathbf{e}^{p^m})^T = f_H(\mathbf{e}, \mathbf{e}^{p^m}) = 0$ and $\mathbf{e}H(\mathbf{e}^{p^{2i-1}})^T = f_H(\mathbf{e}, \mathbf{e}^{p^{2i-1}}) = 0$ for $1 \leq i \leq t$. By rising all entries of $\mathbf{e}H(\mathbf{e}^{p^{2i-1}})^T$ to their $p^{2m-2i+1}$ -th power, we have

$$\mathbf{e}^{p^{2m-2i+1}} H^{p^{2m-2i+1}} \mathbf{e}^T = \mathbf{e}^{p^{2m-2i+1}} H^p \mathbf{e}^T = 0,$$

which gives

$$\mathbf{e}H(\mathbf{e}^{p^{2m-2i+1}})^T = 0.$$

So we obtain $\mathbf{e}H\Psi = (0, 0, \dots, 0)$, where

$$\Psi = ((\mathbf{e}^p)^T, (\mathbf{e}^{p^3})^T, \dots, (\mathbf{e}^{p^m})^T, \dots, (\mathbf{e}^{p^{2m-3}})^T, (\mathbf{e}^{p^{2m-1}})^T).$$

By the choice of \mathbf{e} , it can be shown that Ψ is a nonsingular matrix [9, Corollary 2.38]. Therefore $\mathbf{e}H = (0, 0, \dots, 0)$, which implies that H is a zero matrix. Consequently, φ is injective. On the other hand, a direct calculation shows that $|\mathcal{H}| = |\mathcal{G}| = p^{m^2}$. Hence φ is an isomorphism.

For any $H \in \mathcal{D}$, we have $H = \mathbf{a}^T \mathbf{a}^p$ for some $\mathbf{a} = (a_1, a_2, \dots, a_m) \in W$, where $\mathbf{a}^p = (a_1^p, a_2^p, \dots, a_m^p)$. Therefore,

$$\begin{aligned} \varphi(H) &= (\mathbf{e}H(\mathbf{e}^{p^m})^T, \mathbf{e}H(\mathbf{e}^p)^T, \dots, \mathbf{e}H(\mathbf{e}^{p^{m-2}})^T) \\ &= (\mathbf{e}\mathbf{a}^T \mathbf{a}^p (\mathbf{e}^{p^m})^T, \mathbf{e}\mathbf{a}^T \mathbf{a}^p (\mathbf{e}^p)^T, \dots, \mathbf{e}\mathbf{a}^T \mathbf{a}^p (\mathbf{e}^{p^{m-2}})^T) \\ &= (x^{p^m+1}, x^{p+1}, \dots, x^{p^{m-2}+1}), \end{aligned}$$

where $x = \mathbf{e}\mathbf{a}^T \in \mathbb{F}_q^*$. Thus we obtain $\varphi(\mathcal{D}) \subset \mathcal{S}$. Since

$$|\mathcal{D}| = (p^{2m} - 1)/(p + 1) = (q - 1)/(p + 1) = |\mathcal{S}|,$$

we have $\varphi(\mathcal{D}) = \mathcal{S}$. So we have proved that φ is an isomorphism from \mathcal{H} to \mathcal{G} sending the connection set \mathcal{D} to \mathcal{S} . Therefore Γ_1 is isomorphic to Γ_2 , and they have the same spectrum. ■

From Lemma 4.1 and Lemma 3.2, the eigenvalues of Γ_2 and their multiplicities are known. On the other hand, the eigenvalues of Γ_2 can be expressed using Lemma 3.1.

Note that

$$\widehat{\mathcal{G}} = \{\chi_{(\alpha_0, \alpha_1, \dots, \alpha_t)} \mid \alpha_0 \in \mathbb{F}_{p^m}, \alpha_1, \dots, \alpha_t \in \mathbb{F}_q\},$$

where

$$\chi_{(\alpha_0, \alpha_1, \dots, \alpha_t)}(u) = \zeta_p^{\text{Tr}_1^m(\alpha_0 u_0) + \sum_{j=1}^t \text{Tr}_1^n(\alpha_j u_j)},$$

for any $u = (u_0, u_1, \dots, u_t) \in \mathcal{G}$. By Lemma 3.1, the eigenvalues of Γ_2 are

$$\begin{aligned} &\chi_{(\alpha_0, \alpha_1, \dots, \alpha_t)}(\mathcal{S}) \\ &= \sum_{u \in \mathcal{S}} \zeta_p^{\text{Tr}_1^m(\alpha_0 u_0) + \sum_{j=1}^t \text{Tr}_1^n(\alpha_j u_j)} \\ &= \frac{1}{p+1} \sum_{x \in \mathbb{F}_q^*} \zeta_p^{\text{Tr}_1^m(\alpha_0 x^{p^m+1}) + \sum_{j=1}^t \text{Tr}_1^n(\alpha_j x^{p^{2j-1}+1})} \\ &= \frac{1}{p+1} (T(\alpha_0, \alpha_1, \dots, \alpha_t) - 1), \end{aligned}$$

for all $\alpha_0 \in \mathbb{F}_{p^m}, \alpha_1, \dots, \alpha_t \in \mathbb{F}_q$. Therefore we have

$$T(\alpha_0, \alpha_1, \dots, \alpha_t) = (p+1)\chi_{(\alpha_0, \alpha_1, \dots, \alpha_t)}(\mathcal{S}) + 1.$$

By Lemma 3.2, the eigenvalues of Γ_2 are all rational numbers. Thus, we have $T(\alpha_0, \alpha_1, \dots, \alpha_t) \in \mathbb{Q}$. For any $a \in \mathbb{F}_p^*$, there exists an automorphism $\sigma_a \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ with $\sigma_a(\zeta_p) = \zeta_p^a$. Hence

$$\begin{aligned} \sum_{a \in \mathbb{F}_p^*} T(a\alpha_0, a\alpha_1, \dots, a\alpha_t) &= \sum_{a \in \mathbb{F}_p^*} \sigma_a(T(\alpha_0, \alpha_1, \dots, \alpha_t)) \\ &= (p-1)T(\alpha_0, \alpha_1, \dots, \alpha_t). \end{aligned}$$

Consequently, the Hamming weight of $\mathbf{c}_{[\alpha_0, \alpha_1, \dots, \alpha_t]}$ is

$$\begin{aligned} w_H(\mathbf{c}) &= p^{n-1}(p-1) - \frac{1}{p} \sum_{a \in \mathbb{F}_q^*} T(a\alpha_0, a\alpha_1, \dots, a\alpha_t) \\ &= p^{n-1}(p-1) - \frac{p-1}{p} T(\alpha_0, \alpha_1, \dots, \alpha_t) \\ &= p^{n-1}(p-1) - \frac{p-1}{p} (1 + (p+1)\chi_{(\alpha_0, \alpha_1, \dots, \alpha_t)}(\mathcal{S})). \end{aligned} \quad (1)$$

Now the weight distribution of the code $\mathcal{C}_{(p,m)}$ follows directly from Equation (1) and Lemma 3.2. In the following theorem, we use $[l, k, d]$ code as the notation for a k -dimensional linear code of length l with minimum distance d .

Theorem 4.2: For any odd integer m , the weight distribution of the code $\mathcal{C}_{(p,m)}$ is as follows:

$$A_i = \begin{cases} 1 & \text{if } i = 0, \\ f_j & \text{if } i = w_j, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$w_j = (p^{2m} - p^{2m-1}) \left(1 - \frac{1}{(-p)^j} \right),$$

and

$$f_j = \begin{bmatrix} m \\ j \end{bmatrix}_{(-p)} \prod_{l=0}^{j-1} (p^m - (-p)^l)$$

for $1 \leq j \leq m$. In particular, the code $\mathcal{C}_{(p,m)}$ is a $[p^{2m} - 1, m^2, (p^{2m} - p^{2m-1})(1 - p^{-2})]$ cyclic code.

For a code \mathcal{C} with weight distribution $\{A_0, A_1, \dots, A_l\}$, define its *weight enumerator* as

$$\sum_{i=0}^l A_i x^i.$$

The weight enumerator provides a succinct way to express the weight distribution. For the purpose of illustration, we give two examples below.

Example 4.3: Let $p = 3$, and $m = 3$. The code $\mathcal{C}_{(3,3)}$ is then a $[728, 9, 432]$ code over $\text{GF}(3)$ with the weight enumerator

$$1 + 5460x^{432} + 14040x^{504} + 182x^{648}.$$

Example 4.4: Let $p = 2$, and $m = 5$. The code $\mathcal{C}_{(2,5)}$ is then a $[1023, 25, 384]$ code over $\text{GF}(2)$ with the weight enumerator

$$1 + 57970x^{384} + 12985280x^{480} + 18887680x^{528} \\ + 1623160x^{576} + 341x^{768}.$$

V. CONCLUSION

In the study of cyclic codes, researchers have established the connections between their weight distribution and other mathematical objects, such as Gauss sums (see [6], [17]), algebraic curves (see [20], [22], [25]), as well as quadratic forms (see [5], [10], [11]). In this paper, we found an elegant connection between the weight distribution of a class of cyclic codes and the spectrums of certain distance regular graphs. In this way, the weight distribution of these codes follows from the known spectrums of Hermitian forms graphs. The dual codes of this family of cyclic codes may have arbitrarily many zeros, while most previously known results are obtained in the case where the dual codes have no more than three zeros.

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REFERENCES

- [1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Berlin: Springer-Verlag, 1989, vol. 18.
- [2] P. Delsarte, "On subfield subcodes of modified Reed-Solomon codes," *IEEE Trans. Inform. Theory*, vol. 21, no. 5, pp. 575–576, 1975.
- [3] C. Ding, Y. Liu, C. Ma, and L. Zeng, "The weight distributions of the duals of cyclic codes with two zeros," *IEEE Trans. Inform. Theory*, vol. 57, no. 12, pp. 8000–8006, 2011.
- [4] C. Ding and J. Yang, "Hamming weights in irreducible cyclic codes," *Discrete Math.*, vol. 313, no. 4, pp. 434–446, 2013.
- [5] K. Feng and J. Luo, "Weight distribution of some reducible cyclic codes," *Finite Fields Appl.*, vol. 14, no. 2, pp. 390–409, 2008.
- [6] T. Feng and K. Momihara, "Evaluation of the weight distribution of a class of cyclic codes based on index 2 gauss sums," arXiv: 1206.3354.
- [7] R. Fitzgerald and J. Yucas, "Sums of Gauss sums and weights of irreducible codes," *Finite Fields Appl.*, vol. 11, no. 1, pp. 89–110, 2005.
- [8] H. D. L. Hollmann and Q. Xiang, "On binary cyclic codes with few weights," in *Finite fields and applications (Augsburg, 1999)*. Berlin: Springer, 2001, pp. 251–275.
- [9] R. Lidl and H. Niederreiter, *Finite fields*, ser. Encyclopedia of Mathematics and its Applications. Reading, MA: Addison-Wesley Publishing Company Advanced Book Program, 1983, vol. 20, with a foreword by P. M. Cohn.
- [10] J. Luo and K. Feng, "Cyclic codes and sequences from generalized Coulter-Matthews function," *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5345–5353, 2008.
- [11] —, "On the weight distributions of two classes of cyclic codes," *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5332–5344, 2008.
- [12] J. Luo, Y. Tang, and H. Wang, "On the weight distribution of a class of cyclic codes," in *2009 IEEE International symposium on information theory, Vols 1- 4*, 2009, pp. 1726–1729.
- [13] —, "Cyclic codes and sequences: the generalized Kasami case," *IEEE Trans. Inform. Theory*, vol. 56, no. 5, pp. 2130–2142, 2010.
- [14] C. Ma, L. Zeng, Y. Liu, D. Feng, and C. Ding, "The weight enumerator of a class of cyclic codes," *IEEE Trans. Inform. Theory*, vol. 57, no. 1, pp. 397–402, 2011.
- [15] R. J. McEliece, "Irreducible cyclic codes and Gauss sums," in *Combinatorics (Proc. NATO Advanced Study Inst., Breukelen, 1974), Part 1: Theory of designs, finite geometry and coding theory*. Amsterdam: Math. Centrum, 1974, pp. 179–196. Math. Centre Tracts, No. 55.
- [16] R. J. McEliece and J. H. Rumsey, "Euler products, cyclotomy, and coding," *J. Number Theory*, vol. 4, pp. 302–311, 1972.
- [17] M. Moisisio, "Exponential sums, Gauss sums and cyclic codes," *Acta Univ. Oulu. Ser. A Sci. Rerum Natur.*, no. 306, p. 79, 1998, dissertation, University of Oulu, Oulu, 1998.
- [18] —, "Explicit evaluation of some exponential sums," *Finite Fields Appl.*, vol. 15, no. 6, pp. 644–651, 2009.
- [19] B. Schmidt and C. White, "All two-weight irreducible cyclic codes?" *Finite Fields Appl.*, vol. 8, no. 1, pp. 1–17, 2002.
- [20] R. Schoof, "Families of curves and weight distributions of codes," *Bull. Amer. Math. Soc. (N.S.)*, vol. 32, no. 2, pp. 171–183, 1995. [Online]. Available: <http://dx.doi.org/10.1090/S0273-0979-1995-00586-0>
- [21] D. Stanton, "Three addition theorems for some q -Krawtchouk polynomials," *Geom. Dedicata*, vol. 10, no. 1-4, pp. 403–425, 1981.
- [22] M. van der Vlugt, "Hasse-Davenport curves, Gauss sums, and weight distributions of irreducible cyclic codes," *J. Number Theory*, vol. 55, no. 2, pp. 145–159, 1995.
- [23] G. Vega, "Determining the number of one-weight cyclic codes when length and dimension are given," in *Arithmetic of finite fields*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 2007, vol. 4547, pp. 284–293.
- [24] G. Vega and J. Wolfmann, "New classes of 2-weight cyclic codes," *Des. Codes Cryptogr.*, vol. 42, no. 3, pp. 327–334, 2007.
- [25] B. Wang, C. Tang, Y. Qi, Y. Yang, and M. Xu, "The weight distributions of cyclic codes and elliptic curves," arXiv:1109.0628.
- [26] J. Wolfmann, "Weight distributions of some binary primitive cyclic codes," *IEEE Trans. Inform. Theory*, vol. 40, no. 6, pp. 2068–2071, 1994.
- [27] X. Zeng, N. Li, and L. Hu, "A class of nonbinary codes and sequence families," in *Sequences and their applications—SETA 2008*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 2008, vol. 5203, pp. 81–94.